Oxford Cambridge and RSA

Friday 14 June 2019 – Afternoon

A Level Mathematics A

H240/03 Pure Mathematics and Mechanics

Time allowed: 2 hours

You must have:
- Printed Answer Booklet

You may use:
- a scientific or graphical calculator

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INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total mark for this paper is **100**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **16** pages. The Question Paper consists of **12** pages.

Formulae
A Level Mathematics A (H240)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$$

where ${}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient rule $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Differentiation from first principles

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

Integration

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,dx = \ln|\mathrm{f}(x)| + c$$

$$\int \mathrm{f}'(x)\left(\mathrm{f}(x)\right)^n\,dx = \frac{1}{n+1}\left(\mathrm{f}(x)\right)^{n+1} + c$$

Integration by parts $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A)P(B \mid A) = P(B)P(A \mid B)$ **or** $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

Standard deviation

$\sqrt{\dfrac{\sum(x-\bar{x})^2}{n}} = \sqrt{\dfrac{\sum x^2}{n} - \bar{x}^2}$ **or** $\sqrt{\dfrac{\sum f(x-\bar{x})^2}{\sum f}} = \sqrt{\dfrac{\sum fx^2}{\sum f} - \bar{x}^2}$

The binomial distribution

If $X \sim B(n, p)$ then $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, mean of X is np, variance of X is $np(1-p)$

Hypothesis test for the mean of a normal distribution

If $X \sim N(\mu, \sigma^2)$ then $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1)$

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $P(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

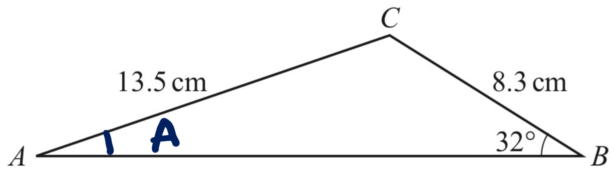
$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Section A: Pure Mathematics

Answer **all** the questions.

1



The diagram shows triangle ABC, with $AC = 13.5\,\text{cm}$, $BC = 8.3\,\text{cm}$ and angle $ABC = 32°$.

Find angle CAB. **[2]**

Using sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b}.$$

$$\frac{\sin A}{8.3} = \frac{\sin 32}{13.5} \Rightarrow \sin A = \frac{8.3 \sin 32}{13.5}$$

$$= \sin A = 0.3258\ldots$$

$$A = \sin^{-1}(\text{ANS})$$

$$= 19.014\ldots$$

$$= 19.0° \ (3sf)$$

2 A circle with centre C has equation $x^2 + y^2 - 6x + 4y + 4 = 0$.

(a) Find

(i) the coordinates of C, [2]

(ii) the radius of the circle. [1]

(b) Determine the set of values of k for which the line $y = kx - 3$ does not intersect or touch the circle. [5]

a) Completing the square

$= x^2 - 6x + y^2 + 4y + 4 = 0$

$(x-3)^2 - (3)^2 + (y+2)^2 - (2)^2 + 4 = 0$

$(x-3)^2 + (y+2)^2 - 9 - \cancel{4} + \cancel{4} = 0$

$(x-3)^2 + (y+2)^2 = 9$.

$(x-a)^2 + (y-b)^2 = r^2$

centre $= (a, b)$, radius $= r$.

$\therefore$ centre $= (3, -2)$

ii) radius $= \sqrt{9} = 3$

b) $y = kx - 3$. substitute into original eqn.

$\Rightarrow x^2 - 6x + (kx-3)^2 + 4(kx-3) + 4 = 0$

$\Rightarrow x^2 - 6x + k^2x^2 - 6kx + 9 + 4kx - 12 + 4 = 0$

$\Rightarrow x^2(1+k^2) + x(-6-2k) + 1 = 0$

If the line doesn't touch or intersect the circle $\rightarrow$ $b^2 - 4ac < 0$

$$b^2 - 4ac \Rightarrow \underbrace{(-6-2k)^2 - 4(1+k^2)(1)}_{\text{expanding this gives;}}$$

$$36 + 24k + 4k^2 - 4(1+k^2)$$

$$\Rightarrow 36 + 24k + \cancel{4k^2} - 4 - \cancel{4k^2}$$

$$\Rightarrow 24k + 32 < 0$$

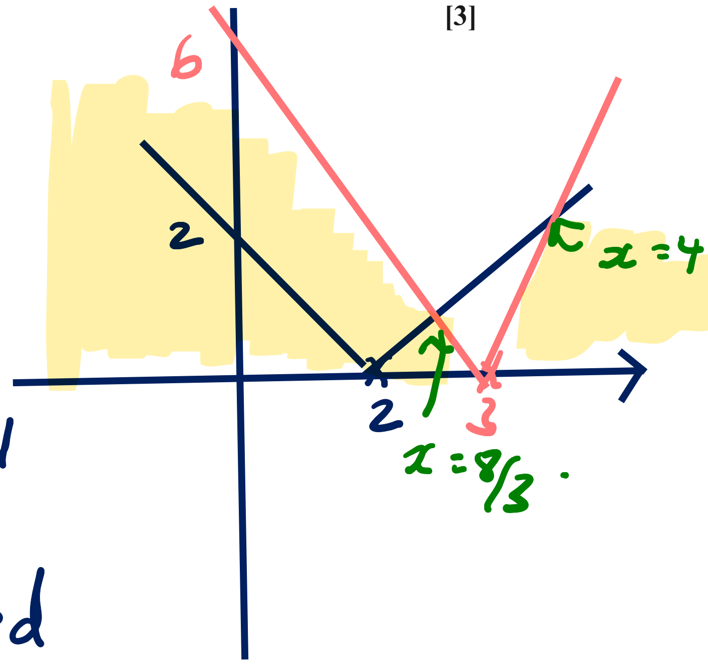
$$k < \frac{-32}{24} \Rightarrow \boxed{k < -\frac{4}{3}}$$

3 (a) In this question you must show detailed reasoning.

Solve the inequality $|x-2| \leqslant |2x-6|$. **[4]**

(b) Give full details of a sequence of two transformations needed to transform the graph of $y=|x-2|$ to the graph of $y=|2x-6|$. **[3]**

a) $\underbrace{|x-2|}_{\text{in Blue}} \leq \underbrace{|2x-6|}_{\text{in red.}}$



→ required region.

Point of intersection of required lines

①

$x-2 = 2x-6$

$\underline{x = 4}$

②

$x-2 = -2x+6$

$3x = 8$

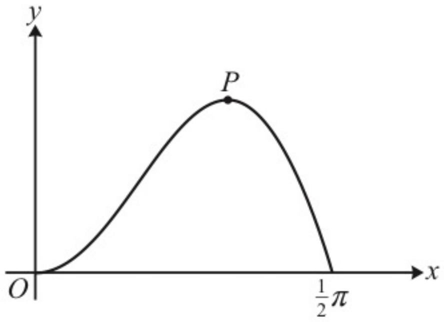
$\underline{x = 8/3}$

$\therefore$ region $= x \leq 8/3$ and $x \geqslant 4$

b) → Translation in positive x-direction by 4 units

→ Stretch by s.f 0.5 in the x-direction.

4



The diagram shows the part of the curve $y = 3x \sin 2x$ for which $0 \leqslant x \leqslant \frac{1}{2}\pi$.

The maximum point on the curve is denoted by P.

(a) Show that the x-coordinate of P satisfies the equation $\tan 2x + 2x = 0$. **[3]**

(b) Use the Newton-Raphson method, with a suitable initial value, to find the x-coordinate of P, giving your answer correct to 4 decimal places. Show the result of each iteration. **[4]**

(c) The trapezium rule, with four strips of equal width, is used to find an approximation to

$$\int_0^{\frac{1}{2}\pi} 3x \sin 2x \, dx.$$

Show that the result can be expressed as $k\pi^2(\sqrt{2}+1)$, where k is a rational number to be determined. **[4]**

(d) **(i)** Evaluate $\int_0^{\frac{1}{2}\pi} 3x \sin 2x \, dx$. **[1]**

(ii) Hence determine whether using the trapezium rule with four strips of equal width gives an under- or over-estimate for the area of the region enclosed by the curve $y = 3x \sin 2x$ and the x-axis for $0 \leqslant x \leqslant \frac{1}{2}\pi$. **[1]**

(iii) Explain briefly why it is not easy to tell from the diagram alone whether the trapezium rule with four strips of equal width gives an under- or over-estimate for the area of the region in this case. **[1]**

a) $y = 3x \sin 2x$

$\frac{dy}{dx} = 3x \cdot 2\cos 2x + 3\sin 2x$ } Using chain rule

$= 6x\cos 2x + 3\sin 2x$.

stationary points occur when $\frac{dy}{dx} = 0$

$$\therefore \frac{6x\cancel{\cos 2x}}{\cancel{\cos 2x}} + \frac{3\sin 2x}{\cos 2x} = \frac{0}{\cos 2x}$$

$$\Rightarrow \frac{3\tan 2x + 6x}{3} = \frac{0}{3}$$

$$\Rightarrow \tan 2x + 2x = 0 \quad \text{as require}$$

b) $f(x) = \tan 2x + 2x$

$$f'(x) = 2\sec^2 2x + 2$$

$$x_{n+1} = x_n - \frac{(\tan 2x_n + 2x_n)}{2\sec^2 2x_n + 2}$$

x_0	x_1	x_2
0.8	0.81389959…	0.839614100…
0.9	0.96102142…	1.00372767…
1.0	1.01365728…	1.01437714…
$\pi/3$	1.01096312…	1.01433905…
1.1	0.99373628…	1.01287398…
1.2	0.93865044…	0.99215748…
1.3	0.87695335…	0.93545928…
1.4	0.82520962…	0.85941601…
1.5	0.79282139…	0.80006720…
1.6	0.78677182…	0.78813953…
1.7	0.81484089…	0.84130177…
1.8	0.88770903…	0.94804414…

$\therefore x = 1.0144$ (4dp)

c) If its 4 equal strips;

$\frac{\pi}{2} \div 4 = \frac{\pi}{8} \rightarrow$ height of each strip.

x	0	$\pi/8$	$\pi/4$	$\pi/2$
$f(x)$	0	$3(\pi/8)\sin(\pi/4)$	$3(\pi/4)\sin(\pi/2)$	$3(\pi/2)\sin(\pi)$
	1	2	3	4

$$\Rightarrow \frac{1}{2}(\pi/8)\left[① + ④ + 2[② + ③] \right]$$

where ①, ②, ③ & ④ are the values of $f(x)$ from the table above

$$\Rightarrow \frac{1}{16}\pi\left[\frac{3}{8}\pi\sqrt{2} + \frac{3}{2}\pi + \frac{9}{8}\pi\sqrt{2}\right]$$

$$\Rightarrow \frac{3}{32}\pi^2\left[\sqrt{2}+1\right] \quad \text{and } k = \frac{3}{32}$$

d) i) $\int_0^{\pi/2} 3x\sin 2x \, dx = \frac{3}{4}\pi$

ii) $\frac{3}{32}\pi^2(\sqrt{2}+1) = 2.23 < 3/4\pi\,(2.356)$

$\therefore$ Trapezium rule gives an under-estimate

iii) The left hand trapezium is above the curve, but the others are below the curve
$\therefore$ overall approximation is not clear.

5 **In this question you must show detailed reasoning.**

(a) Prove that $(\cot\theta + \operatorname{cosec}\theta)^2 = \frac{1+\cos\theta}{1-\cos\theta}$. **[4]**

(b) Hence solve, for $0 < \theta < 2\pi$, $3(\cot\theta + \operatorname{cosec}\theta)^2 = 2\sec\theta$. **[5]**

a) LHS

$(\cot\theta + \operatorname{cosec}\theta)^2 = \cot^2\theta + \operatorname{cosec}^2\theta + 2\cot\theta\operatorname{cosec}\theta$.

$\cot\theta = \frac{\cos\theta}{\sin\theta}$ $\quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$

$\Rightarrow \frac{\cos^2\theta}{\sin^2\theta} + \frac{1}{\sin^2\theta} + \frac{2\cos\theta}{\sin\theta}\left[\frac{1}{\sin\theta}\right]$

$\Rightarrow \frac{\cos^2\theta + 1 + 2\cos\theta\ ]}{\sin^2\theta}$

Numerator

$(1+\cos\theta)^2 \equiv 1 + 2\cos\theta + \cos^2\theta$

$\therefore \frac{(1+\cos\theta)^2}{\sin^2\theta}$

$\sin^2\theta + \cos^2\theta = 1$

$\therefore \sin^2\theta = 1 - \cos^2\theta$

$$\Rightarrow \frac{(1+\cos\theta)^2}{\underbrace{1-\cos^2\theta}_{\text{difference between 2 squares}}} = \frac{(1+\cos\theta)\cancel{(1+\cos\theta)}}{(1-\cos\theta)\cancel{(1+\cos\theta)}}$$

$$\therefore \Rightarrow \frac{1+\cos\theta}{1-\cos\theta} \quad \text{as required.}$$

b) $$3\left[\frac{1+\cos\theta}{1-\cos\theta}\right] = 2\sec\theta \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$3\left[\frac{1+\cos\theta}{1-\cos\theta}\right] = \frac{2}{\cos\theta}$$

$$3\cos\theta\left[\frac{1+\cos\theta}{1-\cos\theta}\right] = 2.$$

$$3\cos\theta + 3\cos^2\theta = 2[1-\cos\theta]$$

$$\Rightarrow 3\cos\theta + 3\cos^2\theta = 2 - 2\cos\theta$$

$$\Rightarrow 3\cos^2\theta + 5\cos\theta - 2 = 0$$

let $\cos\theta = a$.

$$3a^2 + 5a - 2 = 0$$

$$\frac{-5 \pm \sqrt{(-5)^2 - 4(3\times -2)}}{2\times 3}$$

$a = 1/3, -2$

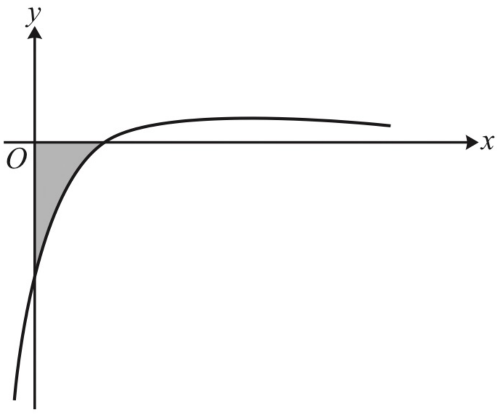
$\therefore \cos\theta = 1/3, -2$

$\cos\theta = 1/3$	$\cos\theta = -2$
$\theta = \cos^{-1}(1/3)$	$-1 \leq \cos\theta \leq 1$
$\theta = 1.23,$ $2\pi - 1.23$	$\therefore$ only answers are
$= 1.23, 5.05$	$\theta = 1.23, 5.05$

S | A ✓

T | C ✓

6



The diagram shows part of the curve $y = \dfrac{2x-1}{(2x+3)(x+1)^2}$.

Find the exact area of the shaded region, giving your answer in the form $p + q\ln r$, where p and q are positive integers and r is a positive rational number. **[10]**

Need to form partial Fractions

$$\frac{2x-1}{(2x+3)(x+1)^2} = \frac{A}{2x+3} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$A(x+1)^2 + B(2x+3)(x+1) + C(2x+3) = 2x-1$$

Let $x = -1$

$C(-1) = 2(-1) - 1 \Rightarrow C = -3$ $\quad$ $C = -3$

Let $x = -3/2$

$$A(-3/2+1)^2 = 2(-3/2) - 1$$

$\frac{A}{4} = -4$ $\quad$ $A = -16$

let $x = 0$

$A(0+1)^2 + B(0+3)(0+1) + C(0+3) = -1$

$A + 3B + 3C = -1$

$-16 + 3B - 9 = -1$

$3B = 24$

$\underline{\underline{B = 8}}$

$\therefore \Rightarrow \dfrac{-16}{2x+3} + \dfrac{8}{x+1} - \dfrac{3}{(x+1)^2}$

$\displaystyle\int \left(\frac{-16}{2x+3} + \frac{8}{x+1} - \frac{3}{(x+1)^2} \right) dx.$

$= -\frac{16}{2} \ln(2x+3) + 8\ln(x+1) + 3(x+1)^{-1} + C$

$= -8\ln(2x+3) + 8\ln(x+1) + 3(x+1)^{-1}$

$a = -8 \quad b = 8 \quad c = 3$

<u>X intercept</u>

$0 = \dfrac{2x-1}{(2x+3)(x+1)^2}$

$2x - 1 = 0$

$x = 1/2$

$$\left[-8\ln(2x+3)+8\ln(x+1)+3(x+1)^{-1}\right]_0^{1/2}$$

$$\left[-8\ln(4)+8\ln\left(\tfrac{3}{2}\right)+3\left(\tfrac{3}{2}\right)^{-1}\right]-$$

$$\left[-8\ln(3)+8\ln(1)+3(1)^{-1}\right]$$

$$\Rightarrow -8\ln 4+8\ln\frac{3}{2}+2+8\ln 3-3.$$

$$\Rightarrow -8\ln\left[\frac{3/2\times 3}{4}\right]-1$$

$$\Rightarrow 1+8\ln 8/9 \qquad p=1 \quad q=-8$$

(as area is positive) $\qquad r=8/9$

Section B: Mechanics

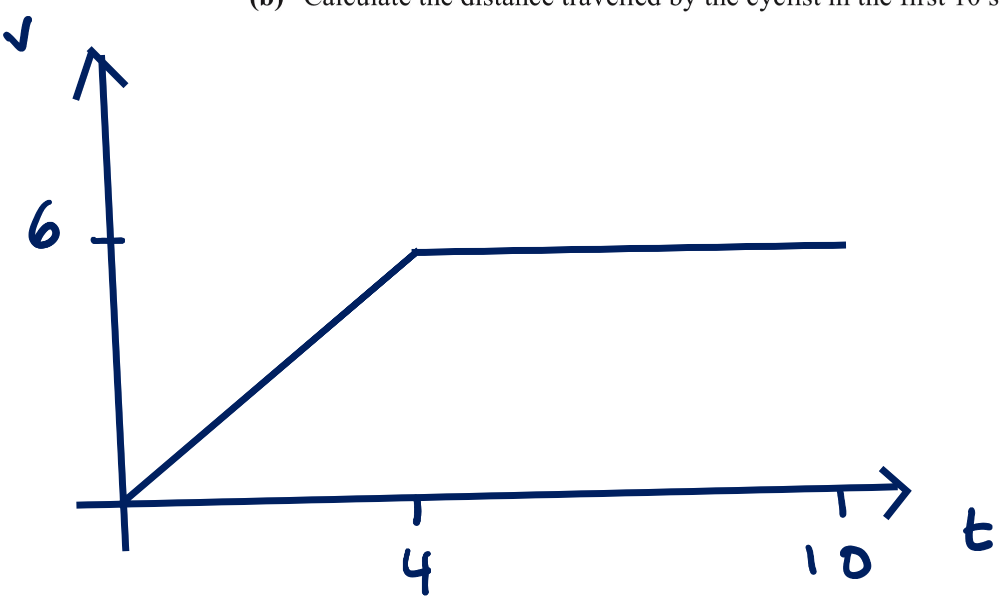
Answer **all** the questions.

7 A cyclist starting from rest accelerates uniformly at $1.5\,\mathrm{m\,s^{-2}}$ for $4\,\mathrm{s}$ and then travels at constant speed.

(a) Sketch a velocity-time graph to represent the first 10 seconds of the cyclist's motion. **[2]**

(b) Calculate the distance travelled by the cyclist in the first 10 seconds. **[2]**

a)



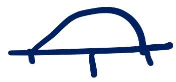
b) Area under the graph.

Area of trapezium $= \frac{1}{2} h (a+b)$

$= \frac{1}{2} \times 6 \times (10+6)$

$= \boxed{48\,\mathrm{m}}$

8 A particle P projected from a point O on horizontal ground hits the ground after 2.4 seconds.



The horizontal component of the initial velocity of P is $\frac{5}{3}d\,\mathrm{m\,s^{-1}}$.

(a) Find, in terms of d, the horizontal distance of P from O when it hits the ground. **[1]**

(b) Find the vertical component of the initial velocity of P. **[2]**

P just clears a vertical wall which is situated at a horizontal distance d m from O.

(c) Find the height of the wall. **[3]**

The speed of P as it passes over the wall is $16\,\mathrm{m\,s^{-1}}$.

(d) Find the value of d correct to 3 significant figures. **[4]**

a) distance = speed × time

$= \frac{5}{3}d \times 2.4 = \underline{\underline{4d}}$

b) $s=0$
$u=u$ ✓
$a=-g$
$t=2.4$

$s = ut + \frac{1}{2}at^2$

$g=9.8$

$0 = u(2.4) - \frac{1}{2}g(2.4)^2$

$\frac{1}{2}g(2.4)^{\cancel{2}\,1} = \cancel{2}.4u$

$u = 11.76\,\mathrm{ms^{-1}}$

c) At horizontal distance dm, what is t?

$d = \frac{5}{3}d \times t$ $\quad t = \underline{\underline{0.6s}}$

$s = ?$
$v = 11.76$
$a = -g$
$t = 0.6$

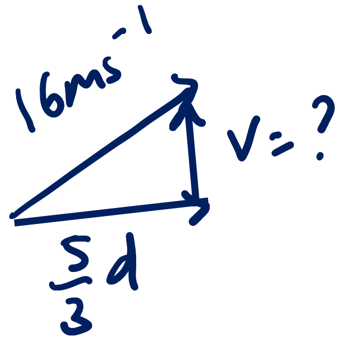
$g = 9.8$

$s = ut + \frac{1}{2}at^2$

$s = 11.76(0.6) - \frac{1}{2}g(0.6)^2$

$s = 5.292\,m$

d)



Finding the vertical component of velocity.

$v = u + at$

$v = 11.76 - g(0.6)$

$v = 5.88$

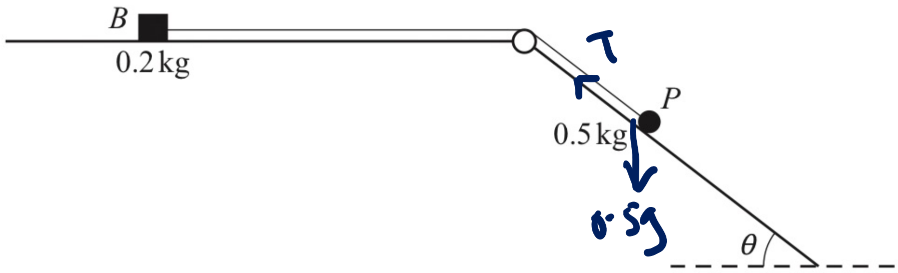
$$\sqrt{(5/3d)^2 + (5.88)^2} = 16$$

$$\frac{25d^2}{9} + 34.5744 = 256$$

$$\frac{25d^2}{9} = 221.4256 \Rightarrow d^2 = 79.71\ldots$$

$d = 8.93$

9



The diagram shows a small block B, of mass 0.2 kg, and a particle P, of mass 0.5 kg, which are attached to the ends of a light inextensible string. The string is taut and passes over a small smooth pulley fixed at the intersection of a horizontal surface and an inclined plane.

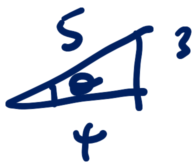
The block can move on the horizontal surface, which is rough. The particle can move on the inclined plane, which is smooth and which makes an angle of θ with the horizontal where $\tan\theta = \frac{3}{4}$.

The system is released from rest. In the first 0.4 seconds of the motion P moves 0.3 m down the plane and B does not reach the pulley.

(a) Find the tension in the string during the first 0.4 seconds of the motion. **[4]**

(b) Calculate the coefficient of friction between B and the horizontal surface. **[5]**

a) $\tan\theta = \frac{3}{4}$ $\quad \sin\theta = \frac{3}{5}$ $\quad \cos\theta = \frac{4}{5}$



a) $s = ut + \frac{1}{2}at^2$

$0.3 = 0 + \frac{1}{2}(a)(0.4)^2$ $\qquad a = 3.75\,ms^{-2}$

Resolving (↘)

$0.5g\sin\theta - T = 0.5a$

$0.5g(3/5) - T = 0.5(3.75)$ $\qquad T = 1.065\,N$

$0.5 \times 9.8 \times 0.6 - 0.5 \times 3.75 = T.$

Resultant force $= 0.2g$.

$F = \mu R$

$\therefore \quad T - F = ma$

$T - \mu R = 0.2(3.75)$

$1.065 - 0.2(9.8)\mu = 0.2(3.75)$

$1.065 - 0.2(3.75) = 0.2(9.8)\mu$

$\mu = \dfrac{0.315}{0.2 \times 9.8} = 0.161$

10 In this question the unit vectors **i** and **j** are in the directions east and north respectively.

A particle R of mass 2 kg is moving on a smooth horizontal surface under the action of a single horizontal force $\mathbf{F}$ N. At time t seconds, the velocity $\mathbf{v}$ m s^{-1} of R, relative to a fixed origin O, is given by $\mathbf{v} = (pt^2 - 3t)\mathbf{i} + (8t + q)\mathbf{j}$, where p and q are constants and $p < 0$.

(a) Given that when $t = 0.5$ the magnitude of $\mathbf{F}$ is 20, find the value of p. **[6]**

When $t = 0$, R is at the point with position vector $(2\mathbf{i} - 3\mathbf{j})$ m.

(b) Find, in terms of q, an expression for the displacement vector of R at time t. **[4]**

When $t = 1$, R is at a point on the line L, where L passes through O and the point with position vector $2\mathbf{i} - 8\mathbf{j}$.

(c) Find the value of q. **[3]**

10a) $\frac{dv}{dt}$ = acceleration.

$v = (pt^2 - 3t)i + (8t + q)j$

$\left.\frac{dv}{dt}\right|_{t=0.5} = (2pt - 3)i + (8)j$

$(2p(0.5) - 3)i + 8j \Rightarrow (p-3)i + 8j$

$F = ma$.

$2\begin{bmatrix} p-3 \\ 8 \end{bmatrix} = \begin{pmatrix} 2p-6 \\ 16 \end{pmatrix}$

$$|F| = \sqrt{(2p-6)^2 + (16)^2} = 20$$

$$(2p-6)^2 + 256 = 400$$

$$(2p-6)^2 = 144$$

$$2p - 6 = \pm 12$$

$$2p = 6 \pm 12$$

$$2p = 18, -6$$

$$p = 9, -3$$ Since $p < 0$ $p = -3$ only.

b) $r_0 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$\int v \, dt \to s$

$$\int (-3t^2 - 3t)i + (8t + 9)j \, dt$$

$$\left(-t^3 - \frac{3t^2}{2}\right)i + (4t^2 + qt)j + c = r$$

when $t=0$ $r = 2i - 3j$

$c = 2i - 3j$

$$\therefore r = \left(-t^3 - \frac{3t^2}{2} + 2\right)i + (4t^2 + qt - 3)j$$

c) $t=1$ $r = ?$

$$r = \left(-1 - \frac{3}{2} + 2\right)i + (4 + q - 3)j$$

$$r = \left(-\frac{1}{2}\right)i + (1+q)j$$

$k\ r_1 = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$

$$k\begin{pmatrix} -1/2 \\ 1+q \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$$

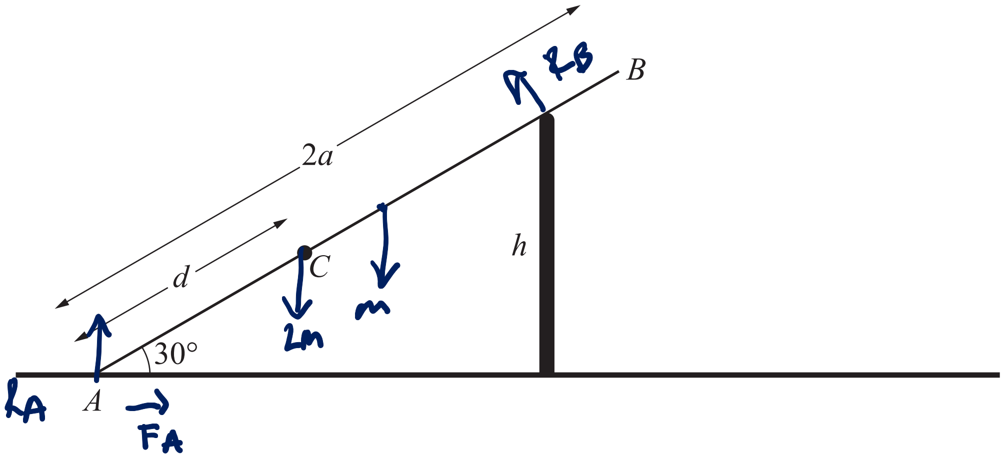
$-\frac{k}{2} = 2$ $\quad \underline{\underline{k = -4}}$ — 1

$k + kq = -8$

$-4 - 4q = -8$

$-4q = -4$ $\quad q = 1$

11



The diagram shows a ladder AB, of length $2a$ and mass m, resting in equilibrium on a vertical wall of height h. The ladder is inclined at an angle of 30° to the horizontal. The end A is in contact with horizontal ground. An object of mass $2m$ is placed on the ladder at a point C where $AC = d$.

The ladder is modelled as uniform, the ground is modelled as being rough, and the vertical wall is modelled as being smooth.

(a) Show that the normal contact force between the ladder and the wall is $\dfrac{mg(a+2d)\sqrt{3}}{4h}$. **[4]**

It is given that the equilibrium is limiting and the coefficient of friction between the ladder and the ground is $\frac{1}{8}\sqrt{3}$.

(b) Show that $h = k(a+2d)$, where k is a constant to be determined. **[7]**

(c) Hence find, in terms of a, the greatest possible value of d. **[2]**

(d) State one improvement that could be made to the model. **[1]**

a) Distance from A to the wall

$\sin 30 = \frac{h}{D}$ $\quad D = \frac{h}{\sin 30}$.

M(A)

$2mgd\cos 30 + mga\cos 30 = R_B \times \frac{h}{\sin 30}$

$[2mgd\cos 30 + mga\cos 30] = R_B h$

$\frac{1}{2h}\left[2mgd \times \frac{\sqrt{3}}{2} + mga \times \frac{\sqrt{3}}{2}\right] = R_B$

$$\Rightarrow \frac{2mgd\sqrt{3} + mga\sqrt{3}}{4h} = R_B = \frac{mg(2d+a)\sqrt{3}}{4h}$$ as required

b) $\mu = \frac{1}{8}\sqrt{3}$

Resolving ($\updownarrow$)

$R_B\cos 30 + R_A = mg + 2mg$

$$R_A = 3mg - \frac{\sqrt{3}}{2}\left[\frac{mg(2d+a)\sqrt{3}}{4h}\right]$$

$F_A = \mu \times R_A$

$F_A = \frac{1}{8}\sqrt{3}\,R_A \quad - (1)$

but Resolving ($\leftrightarrow$)

$F_A = R_B\sin 30 \quad - (2)$

$(1) = (2)$

$$\frac{1}{8}\sqrt{3}\,R_A = \frac{mg(2d+a)\sqrt{3}}{4h} \times \frac{1}{2}$$

$$\frac{1}{8}\sqrt{3} \times \left[3mg - \frac{\sqrt{3}^*}{2}\left(\frac{mg(2d+a)\sqrt{3}^*}{4h}\right)\right] = \frac{mg(2d+a)\sqrt{3}}{8h}.$$

$$\Rightarrow \frac{\sqrt{3}}{8} \times \left[3mg - \frac{3mg(2d+a)}{8h}\right] = \frac{mg(2d+a)\sqrt{3}}{8h}.$$

$$\Rightarrow \frac{3\cancel{\sqrt{3}}\,\cancel{mg}}{\cancel{8}}\left[1 - \frac{(2d+a)}{8h}\right] = \frac{\cancel{\sqrt{3}}\,\cancel{mg}\,(2d+a)}{\cancel{8}h}$$

$$\Rightarrow 3 - \frac{3(2d+a)}{8h} = \frac{2d+a}{h}. \quad \Rightarrow 3 = \frac{2d+a}{h} + \frac{6d+3a}{8h}$$

$$\frac{16d + 8a + 6d + 3a}{8h} = 3.$$

$$22d + 11a = 24h$$

$$\frac{11}{24}(2d + a) = h \qquad \therefore \quad h = \frac{11}{24}(2d+a)$$

$$k = \frac{11}{24}$$

(c) $\frac{11}{24}(2d+a) \leq a$

$$\frac{22d}{24} + \frac{11}{24}a \leq a$$

$$\frac{22d}{24} \leq \frac{13}{24}a$$

$d \leq \frac{13}{22}a$ $\quad \therefore$ greatest value of d is $\frac{13}{22}a$

(d) Any of the following:
→ Model the ladder as non-uniform
→ Include the frictional component for contact of the ladder and the wall
→ consider the thickness of the ladder.